

Sharp Estimates for p -Adic Hardy, Hardy-Littlewood-Pólya Operators and Commutators

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Abstract In this paper we get the sharp estimates of the p -adic Hardy and Hardy-Littlewood-Pólya operators on $L^q(|x|_p^\alpha dx)$. Also, we prove that the commutators generated by the p -adic Hardy operators (Hardy-Littlewood-Pólya operators) and the central BMO functions are bounded on $L^q(|x|_p^\alpha dx)$, more generally, on Herz spaces.

Keywords p -adic Hardy operator, p -adic Hardy-Littlewood-Pólya operator, central BMO function, Herz space.

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1 Introduction

For a prime number p , let \mathbb{Q}_p be the field of p -adic numbers. It is defined as the completion of the field of rational numbers \mathbb{Q} with respect to the non-Archimedean p -adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$; If any non-zero rational number x is represented as $x = p^\gamma \frac{m}{n}$, where m and n are integers which are not divisible by p and γ is an integer, then $|x|_p = p^{-\gamma}$. It is not difficult to show that the norm satisfies the following properties:

$$|xy|_p = |x|_p |y|_p, \quad |x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

It follows from the second property that when $|x|_p \neq |y|_p$, then $|x+y|_p = \max\{|x|_p, |y|_p\}$. From the standard p -adic analysis [28], we see that any non-zero p -adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$x = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \quad \gamma = \gamma(x) \in \mathbb{Z}, \quad (1.1)$$

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where a_j are integers, $0 \leq a_j \leq p-1$, $a_0 \neq 0$. The series (1.1) converges in the p -adic norm because $|a_j p^j|_p = p^{-j}$. Set $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$.

The space \mathbb{Q}_p^n consists of points $x = (x_1, x_2, \dots, x_n)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, \dots, n$. The p -adic norm on \mathbb{Q}_p^n is

$$|x|_p := \max_{1 \leq j \leq n} |x_j|_p, \quad x \in \mathbb{Q}_p^n. \quad (1.2)$$

Denote by

$$B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\gamma\},$$

the ball with center at $a \in \mathbb{Q}_p^n$ and radius p^γ , and its boundary

$$S_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\} = B_\gamma(a) \setminus B_{\gamma-1}(a).$$

Since \mathbb{Q}_p^n is a locally compact commutative group under addition, it follows from the standard analysis that there exists a Haar measure dx on \mathbb{Q}_p^n , which is unique up to positive constant multiple and is translation invariant. We normalize the measure dx by the equality

$$\int_{B_0(0)} dx = |B_0(0)|_H = 1,$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of \mathbb{Q}_p^n . By simple calculation, we can obtain that

$$|B_\gamma(a)|_H = p^{\gamma n}, \quad |S_\gamma(a)|_H = p^{\gamma n}(1 - p^{-n}),$$

for any $a \in \mathbb{Q}_p^n$. For a more complete introduction to the p -adic field, see [28] or [29].

In recent years, p -adic analysis has received a lot of attention due to its application in Mathematical Physics (cf. [1], [3], [18], [19], [27] and [28]). There are numerous papers on p -adic analysis, such as [15] and [16] about Riesz potentials, [2], [5], [7], [22] and [31] about p -adic pseudo-differential equations, etc. The Harmonic Analysis on p -adic field has been drawing more and more concern (cf. [20], [21], [24], [25], [26] and references therein).

The well-known Hardy's integral inequality [14] tells us that for $1 < q < \infty$,

$$\|Hf\|_{L^q(\mathbb{R}^+)} \leq \frac{q}{q-1} \|f\|_{L^q(\mathbb{R}^+)},$$

where the classical Hardy operator is defined by

$$Hf(x) := \frac{1}{x} \int_0^x f(t) dt,$$

for non-negative integral function f on \mathbb{R}^+ , and the constant $\frac{q}{q-1}$ is the best possible. Thus the norm of Hardy operator on $L^q(\mathbb{R}^+)$ is

$$\|H\|_{L^q(\mathbb{R}^+) \rightarrow L^q(\mathbb{R}^+)} = \frac{q}{q-1}.$$

Faris [9] introduced the following n -dimensional Hardy operator, for nonnegative function f on \mathbb{R}^n ,

$$\mathcal{H}f(x) := \frac{1}{\Omega_n |x|^n} \int_{|y| < |x|} f(y) dy, \quad \mathcal{H}^* f(x) := \frac{1}{\Omega_n} \int_{|y| \geq |x|} \frac{f(y)}{|y|^n} dy, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where Ω_n is the volume of the unit ball in \mathbb{R}^n . Christ and Grafakos [6] obtained that the norm of \mathcal{H} and \mathcal{H}^* on $L^q(\mathbb{R}^n)$ are

$$\|\mathcal{H}\|_{L^q(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)} = \|\mathcal{H}^*\|_{L^q(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)} = \frac{q}{q-1},$$

which is the same as that of the 1-dimension Hardy operator. Obviously, \mathcal{H} and \mathcal{H}^* satisfy

$$\int_{\mathbb{R}^n} g(x) \mathcal{H}f(x) dx = \int_{\mathbb{R}^n} f(x) \mathcal{H}^*g(x) dx.$$

In [10], Fu, Grafakos, Lu and Zhao proved that \mathcal{H} is also bounded on the weighted Lebesgue space $L^q(|x|^\alpha dx)$ for $1 < q < \infty$, $\alpha < n(q-1)$. And

$$\|\mathcal{H}\|_{L^q(|x|^\alpha dx) \rightarrow L^q(|x|^\alpha dx)} = \frac{qn}{qn - n - \alpha}.$$

It is clear that when $\alpha = 0$, the result coincides with that we mentioned above. Inspired by these results, in this paper we will introduce the definition of the p -adic Hardy operator and establish the sharp estimates of their boundedness on $L^q(|x|_p^\alpha dx)$.

Definition 1.1. For a function f on \mathbb{Q}_p^n , we define the p -adic Hardy operators as follows

$$\begin{aligned} \mathcal{H}^p f(x) &= \frac{1}{|x|_p^n} \int_{B(0, |x|_p)} f(t) dt, \quad x \in \mathbb{Q}_p^n \setminus \{0\}, \\ \mathcal{H}^{p,*} f(x) &= \int_{\mathbb{Q}_p^n \setminus B(0, |x|_p)} \frac{f(t)}{|t|_p^n} dt, \quad x \in \mathbb{Q}_p^n \setminus \{0\}, \end{aligned} \tag{1.3}$$

where $B(0, |x|_p)$ is a ball in \mathbb{Q}_p^n with center at $0 \in \mathbb{Q}_p^n$ and radius $|x|_p$.

The p -adic Hardy operators \mathcal{H}^p and $\mathcal{H}^{p,*}$ are adjoint mutually:

$$\int_{\mathbb{Q}_p^n} g(x) \mathcal{H}^p f(x) dx = \int_{\mathbb{Q}_p^n} f(x) \mathcal{H}^{p,*} g(x) dx,$$

when $f \in L^q(\mathbb{Q}_p^n)$ and $g \in L^{q'}(\mathbb{Q}_p^n)$, $\frac{1}{q} + \frac{1}{q'} = 1$.

It is obvious that $|\mathcal{H}^p f| \leq \mathcal{M}^p f$, where \mathcal{M}^p is the Hardy-Littlewood maximal operator [20] defined by

$$\mathcal{M}^p f(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma(x)|_H} \int_{B_\gamma(x)} |f(y)| dy, \quad f \in L_{loc}^1(\mathbb{Q}_p^n).$$

The Hardy-Littlewood maximal operator plays a very important role in Harmonic Analysis. The boundedness of \mathcal{M}^p on $L^q(\mathbb{Q}_p^n)$ has been solved (see for instance [29]). But the best estimate of \mathcal{M}^p on $L^q(\mathbb{Q}_p^n)$, $q > 1$, even that of Hardy-Littlewood maximal operator on Euclidean spaces \mathbb{R}^n is very difficult to obtain. Instead, in Section 2, we obtain the sharp estimates of \mathcal{H}^p (and p -adic Hardy-Littlewood-Pólya operator), and the norm of \mathcal{M}^p should be no less than that of \mathcal{H}^p . In Section 3, we will define the commutators of the p -adic Hardy and Hardy-Littlewood-Pólya operators, and discuss the boundedness of them. One of the main innovative points is that we estimate the commutator of Hardy-Littlewood-Pólya operator by that of the p -adic Hardy operator.

2 Sharp estimates of p -adic Hardy and Hardy-Littlewood-Pólya operators

We get the following sharp estimates of \mathcal{H}^p and $\mathcal{H}^{p,*}$ on $L^q(|x|_p^\alpha dx)$.

Theorem 2.1. *Let $1 < q < \infty$ and $\alpha < n(q-1)$. Then*

$$\|\mathcal{H}^p\|_{L^q(|x|_p^\alpha dx) \rightarrow L^q(|x|_p^\alpha dx)} = \|\mathcal{H}^{p,*}\|_{L^q(|x|_p^\alpha dx) \rightarrow L^q(|x|_p^\alpha dx)} = \frac{1 - p^{-n}}{1 - p^{\frac{\alpha}{q} - \frac{n}{q'}}}, \quad (2.1)$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

When $\alpha = 0$, we get the following corollary.

Corollary 2.1. *Let $1 < q < \infty$. Then*

$$\|\mathcal{H}^p\|_{L^q(\mathbb{Q}_p^n) \rightarrow L^q(\mathbb{Q}_p^n)} = \|\mathcal{H}^{p,*}\|_{L^q(\mathbb{Q}_p^n) \rightarrow L^q(\mathbb{Q}_p^n)} = \frac{1 - p^{-n}}{1 - p^{-\frac{n}{q'}}}, \quad (2.2)$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

Remark. *Obviously, the L^q norm of \mathcal{H}^p on \mathbb{Q}_p^n depends on n , however, the L^q norm of \mathcal{H} on \mathbb{R}^n is independent of the dimension n .*

The Hardy-Littlewood-Pólya's linear operator [4] is defined by

$$Tf(x) = \int_0^{+\infty} \frac{f(y)}{\max(x, y)} dy.$$

In [4], the authors obtained that the norm of Hardy-Littlewood-Pólya's operator on $L^q(\mathbb{R}^+)$ (see also P. 254 in [17]), $1 < q < \infty$, is

$$\|T\|_{L^q(\mathbb{R}^+) \rightarrow L^q(\mathbb{R}^+)} = \frac{q^2}{q-1}.$$

Next, we consider p -adic version of Hardy-Littlewood-Pólya operator. We define the p -adic Hardy-Littlewood-Pólya operator as

$$T^p f(x) = \int_{\mathbb{Q}_p^*} \frac{f(y)}{\max(|x|_p, |y|_p)} dy, \quad x \in \mathbb{Q}_p.$$

By the similar method to the proof of Theorem 2.1, we obtain the norm of p -adic Hardy-Littlewood-Pólya operator from $L^q(|x|_p^\alpha dx)$ to $L^q(|x|_p^\alpha dx)$.

Theorem 2.2. *Let $1 < q < \infty$ and $-1 < \alpha < q-1$. Then for any $f \in L^q(|x|_p^\alpha dx)$,*

$$\|T^p f\|_{L^q(|x|_p^\alpha dx)} \leq \left(1 - \frac{1}{p}\right) \left(\frac{1}{1 - p^{\frac{\alpha+1}{q}-1}} + \frac{p^{-\frac{\alpha+1}{q}}}{1 - p^{-\frac{\alpha+1}{q}}} \right) \|f\|_{L^q(|x|_p^\alpha dx)}. \quad (2.3)$$

Moreover,

$$\|T^p\|_{L^q(|x|_p^\alpha dx) \rightarrow L^q(|x|_p^\alpha dx)} = \left(1 - \frac{1}{p}\right) \left(\frac{1}{1 - p^{\frac{\alpha+1}{q}-1}} + \frac{p^{-\frac{\alpha+1}{q}}}{1 - p^{-\frac{\alpha+1}{q}}} \right). \quad (2.4)$$

When $\alpha = 0$, we get the norm of T^p on $L^q(\mathbb{Q}_p)$.

Corollary 2.2. *Let $1 < q < \infty$. Then*

$$\|T^p\|_{L^q(\mathbb{Q}_p) \rightarrow L^q(\mathbb{Q}_p)} = \left(1 - \frac{1}{p}\right) \left(\frac{1}{1 - p^{\frac{1}{q}-1}} + \frac{p^{-\frac{1}{q}}}{1 - p^{-\frac{1}{q}}} \right). \quad (2.5)$$

Proof of Theorem 2.1. Firstly, we claim that the operator \mathcal{H}^p and its restriction to the functions g satisfying $g(x) = g(|x|_p^{-1})$ have the same operator norm on $L^q(|x|_p^\alpha dx)$. In fact, set

$$g(x) = \frac{1}{1 - p^{-n}} \int_{|\xi|_p=1} f(|x|_p^{-1}\xi) d\xi, \quad x \in \mathbb{Q}_p^n.$$

It's easy to see that g satisfies that $g(x) = g(|x|_p^{-1})$ and $\mathcal{H}^p g = \mathcal{H}^p f$. By Hölder's inequality, we get

$$\begin{aligned} \|g\|_{L^q(|x|_p^\alpha dx)} &= \left(\int_{\mathbb{Q}_p^n} \left| \frac{1}{1 - p^{-n}} \int_{|\xi|_p=1} f(|x|_p^{-1}\xi) d\xi \right|^q |x|_p^\alpha dx \right)^{\frac{1}{q}} \\ &\leq \left(\int_{\mathbb{Q}_p^n} \frac{1}{(1 - p^{-n})^q} \left(\int_{|\xi|_p=1} |f(|x|_p^{-1}\xi)|^q d\xi \right) \left(\int_{|\xi|_p=1} d\xi \right)^{\frac{q}{q'}} |x|_p^\alpha dx \right)^{\frac{1}{q}} \\ &= \left(\int_{\mathbb{Q}_p^n} \frac{1}{1 - p^{-n}} \int_{|\xi|_p=1} |f(|x|_p^{-1}\xi)|^q d\xi |x|_p^\alpha dx \right)^{\frac{1}{q}} \\ &= \frac{1}{(1 - p^{-n})^{\frac{1}{q}}} \left(\int_{\mathbb{Q}_p^n} \int_{|y|_p=|x|_p} |f(y)|^q dy |x|_p^{\alpha-n} dx \right)^{\frac{1}{q}} \\ &= \frac{1}{(1 - p^{-n})^{\frac{1}{q}}} \left(\int_{\mathbb{Q}_p^n} \int_{|x|_p=|y|_p} |x|_p^{\alpha-n} dx |f(y)|^q dy \right)^{\frac{1}{q}} \\ &= \|f\|_{L^q(|x|_p^\alpha dx)}. \end{aligned}$$

Therefore,

$$\frac{\|\mathcal{H}^p f\|_{L^q(|x|_p^\alpha dx)}}{\|f\|_{L^q(|x|_p^\alpha dx)}} \leq \frac{\|\mathcal{H}^p g\|_{L^q(|x|_p^\alpha dx)}}{\|g\|_{L^q(|x|_p^\alpha dx)}},$$

which implies that the claim. In the following, without loss of generality, we may

assume that $f \in L^q(|x|_p^\alpha dx)$ satisfies that $f(x) = f(|x|_p^{-1})$.

By changing of variables $t = |x|_p^{-1}y$, we have

$$\begin{aligned}\|\mathcal{H}^p f\|_{L^q(|x|_p^\alpha dx)} &= \left(\int_{\mathbb{Q}_p^n} \left| \frac{1}{|x|_p^n} \int_{B(0, |x|_p)} f(t) dt \right|^q |x|_p^\alpha dx \right)^{\frac{1}{q}} \\ &= \left(\int_{\mathbb{Q}_p^n} \left| \int_{B(0,1)} f(|x|_p^{-1}y) dy \right|^q |x|_p^\alpha dx \right)^{\frac{1}{q}}\end{aligned}$$

Then using Minkowski's integral inequality, we get

$$\begin{aligned}\|\mathcal{H}^p f\|_{L^q(|x|_p^\alpha dx)} &\leq \int_{B(0,1)} \left(\int_{\mathbb{Q}_p^n} |f(|y|_p^{-1}x)|^q |x|_p^\alpha dx \right)^{\frac{1}{q}} dy \\ &\leq \left(\int_{B(0,1)} |y|_p^{-\frac{n}{q}-\frac{\alpha}{q}} dy \right) \|f\|_{L^q(|x|_p^\alpha dx)} \\ &= \left(\sum_{k=0}^{\infty} \int_{|y|_p=p^{-k}} p^{\frac{k(n+\alpha)}{q}} dy \right) \|f\|_{L^q(|x|_p^\alpha dx)} \\ &= \frac{1-p^{-n}}{1-p^{\frac{\alpha}{q}-\frac{n}{q'}}} \|f\|_{L^q(|x|_p^\alpha dx)},\end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Therefore, we get

$$\|\mathcal{H}^p\|_{L^q(|x|_p^\alpha dx) \rightarrow L^q(|x|_p^\alpha dx)} \leq \frac{1-p^{-n}}{1-p^{\frac{\alpha}{q}-\frac{n}{q'}}}. \quad (2.6)$$

On the other hand, for $0 < \epsilon < 1$, we take

$$f_\epsilon = \begin{cases} 0, & |x|_p < 1, \\ |x|_p^{-\frac{n}{q}-\frac{\alpha}{q}-\epsilon}, & |x|_p \geq 1. \end{cases}$$

Then $\|f_\epsilon\|_{L^q(|x|_p^\alpha dx)}^q = \frac{1-p^{-n}}{1-p^{-\epsilon q}}$, and

$$\mathcal{H}^p f_\epsilon(x) = \begin{cases} 0, & |x|_p < 1, \\ |x|_p^{-\frac{n}{q}-\frac{\alpha}{q}-\epsilon} \int_{\frac{1}{|x|_p} \leq |t|_p \leq 1} |t|_p^{-\frac{n}{q}-\frac{\alpha}{q}-\epsilon} dt, & |x|_p \geq 1. \end{cases}$$

Assume $|\epsilon|_p > 1$. We have

$$\begin{aligned}\|\mathcal{H}^p f_\epsilon\|_{L^q(|x|_p^\alpha dx)} &= \left\{ \int_{|x|_p \geq 1} \left(|x|_p^{-\frac{n}{q}-\frac{\alpha}{q}-\epsilon} \int_{\frac{1}{|x|_p} \leq |t|_p \leq 1} |t|_p^{-\frac{n}{q}-\frac{\alpha}{q}-\epsilon} dt \right)^q |x|_p^\alpha dx \right\}^{\frac{1}{q}} \\ &\geq \left\{ \int_{|x|_p \geq |\epsilon|_p} \left(|x|_p^{-\frac{n}{q}-\frac{\alpha}{q}-\epsilon} \int_{\frac{1}{|\epsilon|_p} \leq |t|_p \leq 1} |t|_p^{-\frac{n}{q}-\frac{\alpha}{q}-\epsilon} dt \right)^q |x|_p^\alpha dx \right\}^{\frac{1}{q}} \quad (2.7) \\ &= \left(\int_{|x|_p \geq |\epsilon|_p} |x|_p^{-n-\epsilon q} dx \right)^{\frac{1}{q}} \int_{\frac{1}{|\epsilon|_p} \leq |t|_p \leq 1} |t|_p^{-\frac{n}{q}-\frac{\alpha}{q}-\epsilon} dt\end{aligned}$$

$$= \|f_\epsilon\|_{L^q(|x|_p^\alpha dx)} |\epsilon|_p^{-\epsilon} \int_{\frac{1}{|\epsilon|_p} \leq |t|_p \leq 1} |t|_p^{-\frac{n}{q} - \frac{\alpha}{q} - \epsilon} dt.$$

Therefore,

$$\int_{\frac{1}{|\epsilon|_p} \leq |t|_p \leq 1} |t|_p^{-\frac{n}{q} - \frac{\alpha}{q} - \epsilon} dt \leq \|\mathcal{H}^p\|_{L^q(|x|_p^\alpha dx) \rightarrow L^q(|x|_p^\alpha dx)} |\epsilon|_p^\epsilon. \quad (2.8)$$

Now take $\epsilon = p^{-k}$, $k = 1, 2, 3, \dots$. Then $|\epsilon|_p = p^k > 1$. Letting k approach to ∞ , then ϵ approaches to 0 and $|\epsilon|_p^\epsilon = p^{\frac{k}{p^k}}$ approaches to 1. Then by Fatou's Lemma, we obtain

$$\frac{1 - p^{-n}}{1 - p^{\frac{\alpha}{q} - \frac{n}{q'}}} = \int_{0 < |t|_p \leq 1} |t|_p^{-\frac{n}{q} - \frac{\alpha}{q}} dt \leq \|\mathcal{H}^p\|_{L^q(|x|_p^\alpha dx) \rightarrow L^q(|x|_p^\alpha dx)}. \quad (2.9)$$

Then (2.6) and (2.9) imply that

$$\|\mathcal{H}^p\|_{L^q(|x|_p^\alpha dx) \rightarrow L^q(|x|_p^\alpha dx)} = \frac{1 - p^{-n}}{1 - p^{\frac{\alpha}{q} - \frac{n}{q'}}}.$$

By the similar method, we can also obtain that

$$\|\mathcal{H}^{p,*}\|_{L^q(|x|_p^\alpha dx) \rightarrow L^q(|x|_p^\alpha dx)} = \frac{1 - p^{-n}}{1 - p^{\frac{\alpha}{q} - \frac{n}{q'}}}.$$

Theorem 2.1 is proved. \square

Proof of Theorem 2.2. By definition and the change of variables $y = xt$, we have

$$\begin{aligned} \|T^p f\|_{L^q(|x|_p^\alpha dx)} &= \left(\int_{\mathbb{Q}_p} \left| \int_{\mathbb{Q}_p^*} \frac{f(y)}{\max(|x|_p, |y|_p)} dy \right|^q |x|_p^\alpha dx \right)^{\frac{1}{q}} \\ &\leq \left(\int_{\mathbb{Q}_p} \left(\int_{\mathbb{Q}_p^*} \frac{|f(y)|}{\max(|x|_p, |y|_p)} dy \right)^q |x|_p^\alpha dx \right)^{\frac{1}{q}} \\ &= \left(\int_{\mathbb{Q}_p} \left(\int_{\mathbb{Q}_p^*} \frac{|f(xt)|}{\max(1, |t|_p)} dt \right)^q |x|_p^\alpha dx \right)^{\frac{1}{q}}. \end{aligned}$$

By Minkowski's integral inequality, we get

$$\begin{aligned} \|T^p f\|_{L^q(|x|_p^\alpha dx)} &\leq \int_{\mathbb{Q}_p^*} \left(\int_{\mathbb{Q}_p} |f(xt)|^q |x|_p^\alpha dx \right)^{\frac{1}{q}} \frac{1}{\max(1, |t|_p)} dt \\ &\leq \int_{\mathbb{Q}_p^*} \left(\int_{\mathbb{Q}_p} |f(x)|^q |x|_p^\alpha dx \right)^{\frac{1}{q}} \frac{|t|_p^{-\frac{\alpha+1}{q}}}{\max(1, |t|_p)} dt \\ &= \|f\|_{L^q(|x|_p^\alpha dx)} \int_{\mathbb{Q}_p^*} \frac{|t|_p^{-\frac{\alpha+1}{q}}}{\max(1, |t|_p)} dt. \end{aligned} \quad (2.10)$$

Since

$$\begin{aligned}
\int_{\mathbb{Q}_p^*} \frac{|t|_p^{-\frac{\alpha+1}{q}}}{\max(1, |t|_p)} dt &= \sum_{k=-\infty}^0 \int_{S_k} \frac{|t|_p^{-\frac{\alpha+1}{q}}}{\max(1, |t|_p)} dt + \sum_{k=1}^{+\infty} \int_{S_k} \frac{|t|_p^{-\frac{\alpha+1}{q}}}{\max(1, |t|_p)} dt \\
&= \left(1 - \frac{1}{p}\right) \left(\sum_{k=-\infty}^0 p^{k(1-\frac{\alpha+1}{q})} + \sum_{k=1}^{+\infty} p^{-\frac{k(\alpha+1)}{q}} \right) \\
&= \left(1 - \frac{1}{p}\right) \left(\frac{1}{1 - p^{\frac{\alpha+1}{q}-1}} + \frac{p^{-\frac{\alpha+1}{q}}}{1 - p^{-\frac{\alpha+1}{q}}} \right).
\end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.10) shows that (2.3) holds and

$$\|T^p\|_{L^q(|x|_p^\alpha dx) \rightarrow L^q(|x|_p^\alpha dx)} \leq \left(1 - \frac{1}{p}\right) \left(\frac{1}{1 - p^{\frac{\alpha+1}{q}-1}} + \frac{p^{-\frac{\alpha+1}{q}}}{1 - p^{-\frac{\alpha+1}{q}}} \right). \tag{2.12}$$

On the other hand, for $0 < \epsilon < 1$, let

$$f_\epsilon = \begin{cases} 0, & |x|_p < 1, \\ |x|_p^{\frac{-1-\alpha}{q}-\epsilon}, & |x|_p \geq 1. \end{cases}$$

Then $\|f_\epsilon\|_{L^q(|x|_p^\alpha dx)}^q = \frac{1-p^{-1}}{1-p^{-\epsilon q}}$, and

$$T^p f_\epsilon(x) = \int_{|y|_p \geq 1} \frac{|y|_p^{\frac{-1-\alpha}{q}-\epsilon}}{\max(|x|_p, |y|_p)} dy.$$

Set $|\epsilon|_p > 1$. We have

$$\begin{aligned}
\|T^p f_\epsilon\|_{L^q(|x|_p^\alpha dx)} &= \left\{ \int_{\mathbb{Q}_p} \left(\int_{|y|_p \geq 1} \frac{|y|_p^{\frac{-1-\alpha}{q}-\epsilon}}{\max(|x|_p, |y|_p)} dy \right)^q |x|_p^\alpha dx \right\}^{\frac{1}{q}} \\
&\geq \left\{ \int_{|x|_p \geq |\epsilon|_p} \left(\int_{|t|_p \geq \frac{1}{|\epsilon|_p}} \frac{|t|_p^{\frac{-1-\alpha}{q}-\epsilon}}{\max(1, |t|_p)} dt \right)^q |x|_p^{-1-\epsilon q} dx \right\}^{\frac{1}{q}} \\
&= \|f_\epsilon\|_{L^q(|x|_p^\alpha dx)}^q |\epsilon|_p^{-\epsilon} \int_{|t|_p \geq \frac{1}{|\epsilon|_p}} \frac{|t|_p^{\frac{-1-\alpha}{q}-\epsilon}}{\max(1, |t|_p)} dt.
\end{aligned} \tag{2.13}$$

Therefore,

$$\int_{|t|_p \geq \frac{1}{|\epsilon|_p}} \frac{|t|_p^{\frac{-1-\alpha}{q}-\epsilon}}{\max(1, |t|_p)} dt \leq \|T^p\|_{L^q(|x|_p^\alpha dx) \rightarrow L^q(|x|_p^\alpha dx)} |\epsilon|_p^\epsilon. \tag{2.14}$$

Now take $\epsilon = p^{-k}$, $k = 1, 2, 3, \dots$. Then $|\epsilon|_p = p^k > 1$. Letting k approach to ∞ and using Fatou's Lemma, we obtain

$$\left(1 - \frac{1}{p}\right) \left(\frac{1}{1 - p^{\frac{\alpha+1}{q}-1}} + \frac{p^{-\frac{\alpha+1}{q}}}{1 - p^{-\frac{\alpha+1}{q}}} \right) = \int_{\mathbb{Q}_p^*} \frac{|t|_p^{-\frac{\alpha+1}{q}}}{\max(1, |t|_p)} dt \quad (2.15)$$

$$\leq \|T^p\|_{L^q(|x|_p^\alpha dx) \rightarrow L^q(|x|_p^\alpha dx)}.$$

Then (2.4) follows from (2.12) and (2.15). Theorem 2.2 is proved. \square

3 Boundedness of commutators of p -adic Hardy and Hardy-Littlewood-Pólya operators

The boundedness of commutators is an active topic in harmonic analysis due to its important applications, for example, it can be applied to characterizing some function spaces. There are a lot of works about the boundedness of commutators of various Hardy-type operators on Euclidean spaces (cf. [11], [12]). In this section, we consider the boundedness of commutators of p -adic Hardy and Hardy-Littlewood-Pólya operators.

Definition 3.1. Let $b \in L_{loc}(\mathbb{Q}_p^n)$. The commutators of p -adic Hardy operators are defined by

$$\mathcal{H}_b^p f = b\mathcal{H}^p f - \mathcal{H}^p(bf), \quad \mathcal{H}_b^{p,*} f = b\mathcal{H}^{p,*} f - \mathcal{H}^{p,*}(bf). \quad (3.1)$$

Definition 3.2. Let $b \in L_{loc}(\mathbb{Q}_p^n)$. The commutator of p -adic Hardy-Littlewood-Pólya operator is defined by

$$T_b^p f = bT^p f - T^p(bf). \quad (3.2)$$

In [8], [13] and [23], the CMO spaces (central BMO spaces) on \mathbb{R}^n have been introduced and studied. CMO spaces bears a simple relationship with BMO: $g \in BMO$ precisely when g and all of its translates belong to BMO spaces uniformly a.e.. Many precise analogies exist between CMO spaces and BMO space from the point of view of real Hardy spaces. Similarly, we define the CMO^q spaces on \mathbb{Q}_p^n .

Definition 3.3. Let $1 \leq q < \infty$. A function $f \in L_{loc}^q(\mathbb{Q}_p^n)$ is said to be in $CMO^q(\mathbb{Q}_p^n)$, if

$$\|f\|_{CMO^q(\mathbb{Q}_p^n)} := \sup_{\gamma \in \mathbb{Z}} \left(\frac{1}{|B_\gamma(0)|_H} \int_{B_\gamma(0)} |f(x) - f_{B_\gamma(0)}|^q dx \right)^{\frac{1}{q}} < \infty,$$

where

$$f_{B_\gamma(0)} = \frac{1}{|B_\gamma(0)|_H} \int_{B_\gamma(0)} f(x) dx.$$

Remark 3.1. *It is obvious that $L^\infty(\mathbb{Q}_p^n) \subset BMO(\mathbb{Q}_p^n) \subset CMO^q(\mathbb{Q}_p^n)$.*

Since Herz space is a natural generalization of Lebesgue space with power weight, we further study the boundedness of commutators of p -adic Hardy and Hardy-Littlewood-Pólya operators on Herz space. Let us first give the definition of Herz space.

Let $B_k = B_k(0) = \{x \in \mathbb{Q}_p^n : |x|_p \leq p^k\}$, $S_k = B_k \setminus B_{k-1}$ and χ_E is the characteristic function of set E .

Definition 3.4. [30] *Suppose $\alpha \in \mathbb{R}$, $0 < q < \infty$ and $0 < r < \infty$. The homogeneous p -adic Herz space $K_r^{\alpha,q}(\mathbb{Q}_p^n)$ is defined by*

$$K_r^{\alpha,q}(\mathbb{Q}_p^n) = \left\{ f \in L_{loc}^r(\mathbb{Q}_p^n) : \|f\|_{K_r^{\alpha,q}(\mathbb{Q}_p^n)} < \infty \right\},$$

where

$$\|f\|_{K_r^{\alpha,q}(\mathbb{Q}_p^n)} = \left(\sum_{k=-\infty}^{+\infty} p^{k\alpha q} \|f \chi_k\|_{L^r(\mathbb{Q}_p^n)}^q \right)^{\frac{1}{q}},$$

with the usual modifications made when $q = \infty$ or $r = \infty$.

Remark 3.2. $K_q^{0,q}(\mathbb{Q}_p^n)$ is the generalization of $L^q(|x|_p^\alpha dx)$, and $K_q^{\frac{\alpha}{q},q}(\mathbb{Q}_p^n) = L^q(|x|_p^\alpha dx)$, $K_q^{0,q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$ for all $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$.

In what follows, we will not study the best estimates of the two commutators mentioned above, instead, we will discuss the boundedness of them. Motivated by [11], we get the following operator boundedness results. Throughout this paper, we use C to denote different positive constants which are independent of the essential variables, and q' to denote the conjugate index of q , i.e. $\frac{1}{q} + \frac{1}{q'} = 1$.

Theorem 3.1. *Let $0 < q_1 \leq q_2 < \infty$, $1 < r < \infty$ and $b \in CMO^{\max\{q',r\}}(\mathbb{Q}_p^n)$. Then*

- (1) *if $\alpha < \frac{n}{r'}$, then \mathcal{H}_b^p is bounded from $K_r^{\alpha,q_1}(\mathbb{Q}_p^n)$ to $K_r^{\alpha,q_2}(\mathbb{Q}_p^n)$;*
- (2) *if $\alpha > -\frac{n}{r}$, then $\mathcal{H}_b^{p,*}$ is bounded from $K_r^{\alpha,q_1}(\mathbb{Q}_p^n)$ to $K_r^{\alpha,q_2}(\mathbb{Q}_p^n)$.*

From Remark 3.2, we get the following two corollaries.

Corollary 3.1. *Suppose that $1 < q < \infty$ and $b \in CMO^{\max\{q',q\}}(\mathbb{Q}_p^n)$. Then*

- (1) *if $\alpha < \frac{nq}{q'}$, then \mathcal{H}_b^p is bounded from $L^q(|x|_p^\alpha dx)$ to $L^q(|x|_p^\alpha dx)$;*
- (2) *if $\alpha > -n$, then $\mathcal{H}_b^{p,*}$ is bounded from $L^q(|x|_p^\alpha dx)$ to $L^q(|x|_p^\alpha dx)$.*

Corollary 3.2. *Suppose that $1 < q < \infty$ and $b \in CMO^{\max\{q',q\}}(\mathbb{Q}_p^n)$. Then both \mathcal{H}_b^p and $\mathcal{H}_b^{p,*}$ are bounded from $L^q(\mathbb{Q}_p^n)$ to $L^q(\mathbb{Q}_p^n)$.*

Due to Theorem 3.1, we can also obtain the boundedness of commutator generated by p -adic Hardy-Littlewood-Pólya operator and CMO function.

Theorem 3.2. *Suppose that $0 < q_1 \leq q_2 < \infty$, $1 < r < \infty$, $-\frac{1}{r} < \alpha < \frac{1}{r'}$ and $b \in CMO^{\max\{r', r\}}(\mathbb{Q}_p^n)$. Then T_b^p is bounded from $K_r^{\alpha, q_1}(\mathbb{Q}_p^n)$ to $K_r^{\alpha, q_2}(\mathbb{Q}_p^n)$.*

Corollary 3.3. *Suppose that $1 < q < \infty$, $-1 < \alpha < q - 1$ and $b \in CMO^{\max\{q', q\}}(\mathbb{Q}_p^n)$. Then T_b^p and $T_b^{p,*}$ is bounded from $L^q(|x|_p^\alpha dx)$ to $L^q(|x|_p^\alpha dx)$.*

Corollary 3.4. *Suppose that $1 < q < \infty$ and $b \in CMO^{\max\{q', q\}}(\mathbb{Q}_p^n)$. Then T_b^p is bounded from $L^q(\mathbb{Q}_p^n)$ to $L^q(\mathbb{Q}_p^n)$.*

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.1. *Suppose that b is a CMO function and $1 \leq q < r < \infty$. Then $CMO^r(\mathbb{Q}_p^n) \subset CMO^q(\mathbb{Q}_p^n)$ and $\|b\|_{CMO^q(\mathbb{Q}_p^n)} \leq \|b\|_{CMO^r(\mathbb{Q}_p^n)}$.*

Proof. For any $b \in CMO^r(\mathbb{Q}_p^n)$, by Hölder's inequality, we have

$$\begin{aligned} & \left(\frac{1}{|B_\gamma(0)|_H} \int_{B_\gamma(0)} |b(x) - b_{B_\gamma(0)}|^q dx \right)^{\frac{1}{q}} \\ & \leq \left\{ \frac{1}{|B_\gamma(0)|_H} \left(\int_{B_\gamma(0)} |b(x) - b_{B_\gamma(0)}|^{q \frac{r}{r-q}} dx \right)^{\frac{q}{r}} \left(\int_{B_\gamma(0)} 1 dx \right)^{1 - \frac{q}{r}} \right\}^{\frac{1}{q}} \\ & = \left\{ \frac{1}{|B_\gamma(0)|_H} \left(\int_{B_\gamma(0)} |b(x) - b_{B_\gamma(0)}|^r dx \right)^{\frac{q}{r}} |B_\gamma(0)|_H^{1 - \frac{q}{r}} \right\}^{\frac{1}{q}} \\ & = \left(\frac{1}{|B_\gamma(0)|_H} \int_{B_\gamma(0)} |b(x) - b_{B_\gamma(0)}|^r dx \right)^{\frac{1}{r}} \\ & \leq \|b\|_{CMO^r(\mathbb{Q}_p^n)} \end{aligned}$$

Therefore, $b \in CMO^q(\mathbb{Q}_p^n)$ and $\|b\|_{CMO^q(\mathbb{Q}_p^n)} \leq \|b\|_{CMO^r(\mathbb{Q}_p^n)}$. This completes the proof. \square

Lemma 3.2. *Suppose that b is a CMO function, $j, k \in \mathbb{Z}$. Then*

$$|b(t) - b_{B_k}| \leq |b(t) - b_{B_j}| + p^n |j - k| \|b\|_{CMO^1(\mathbb{Q}_p^n)}. \quad (3.3)$$

Proof. For $i \in \mathbb{Z}$, recall that $b_{B_i} = \frac{1}{|B_i|_H} \int_{B_i} b(x) dx$, we have

$$\begin{aligned} |b_{B_i} - b_{B_{i+1}}| &\leq \frac{1}{|B_i|_H} \int_{B_i} |b(t) - b_{B_{i+1}}| dt \leq \frac{p^n}{|B_{i+1}|_H} \int_{B_{i+1}} |b(t) - b_{B_{i+1}}| dt \\ &\leq p^n \|b\|_{CMO^1(\mathbb{Q}_p^n)}. \end{aligned} \quad (3.4)$$

For $j, k \in \mathbb{Z}$, without loss of generality, we can assume that $j \leq k$. By (3.4) we get

$$|b(t) - b_{B_k}| \leq |b(t) - b_{B_j}| + \sum_{i=k}^{j-1} |b_{B_i} - b_{B_{i+1}}| \leq |b(t) - b_{B_j}| + p^n |j - k| \|b\|_{CMO^1(\mathbb{Q}_p^n)}.$$

The lemma is proved. \square

Proof of Theorem 3.1. Denote $f(x)\chi_i(x) = f_i(x)$. By definition

$$\begin{aligned} \|(\mathcal{H}_b^p f)\chi_k\|_{L^r(\mathbb{Q}_p^n)}^r &= \int_{S_k} |x|_p^{-rn} \left| \int_{B(0, |x|_p)} f(t)(b(x) - b(t)) dt \right|^r dx \\ &\leq \int_{S_k} p^{-krn} \left(\int_{B(0, p^k)} |f(t)(b(x) - b(t))| dt \right)^r dx \\ &= p^{-krn} \int_{S_k} \left(\sum_{j=-\infty}^k \int_{S_j} |f(t)(b(x) - b(t))| dt \right)^r dx \\ &\leq Cp^{-krn} \int_{S_k} \left(\sum_{j=-\infty}^k \int_{S_j} |f(t)(b(x) - b_{B_k})| dt \right)^r dx \\ &\quad Cp^{-krn} \int_{S_k} \left(\sum_{j=-\infty}^k \int_{S_j} |f(t)(b(t) - b_{B_k})| dt \right)^r dx \\ &:= I + II. \end{aligned}$$

Now let's estimate I and J , respectively. For I , by Hölder's inequality ($\frac{1}{r} + \frac{1}{r'} = 1$), we have

$$\begin{aligned} I &= Cp^{-krn} \left(\int_{S_k} |b(x) - b_{B_k}|^r dx \right) \left(\sum_{j=-\infty}^k \int_{S_j} |f(t)| dt \right)^r \\ &\leq Cp^{-\frac{krn}{r'}} \left(\frac{1}{|B_k|_H} \int_{B_k} |b(x) - b_{B_k}|^r dx \right) \times \\ &\quad \left\{ \sum_{j=-\infty}^k \left(\int_{S_j} |f(t)|^r dt \right)^{\frac{1}{r}} \left(\int_{S_j} dt \right)^{\frac{1}{r'}} \right\}^r \end{aligned} \quad (3.5)$$

$$\leq C \|b\|_{CMO^r(\mathbb{Q}_p^n)}^r \left\{ \sum_{j=-\infty}^k p^{\frac{(j-k)n}{r'}} \|f_j\|_{L^r(\mathbb{Q}_p^n)} \right\}^r.$$

For II , by Lemma 3.2, we get

$$\begin{aligned} II &= C p^{-krn} \int_{S_k} \left(\sum_{j=-\infty}^k \int_{S_j} |f(t)(b(t) - b_{B_k})| dt \right)^r dx \\ &= C p^{-krn} p^{kn} (1 - p^{-n}) \left(\sum_{j=-\infty}^k \int_{S_j} |f(t)(b(t) - b_{B_k})| dt \right)^r \\ &\leq C p^{\frac{-krn}{r'}} \left(\sum_{j=-\infty}^k \int_{S_j} |f(t)(b(t) - b_{B_j})| dt \right)^r \\ &\quad + C p^{\frac{-krn}{r'}} \|b\|_{CMO^1(\mathbb{Q}_p^n)}^r \left(\sum_{j=-\infty}^k (k-j) \int_{S_j} |f(t)| dt \right)^r \\ &:= II_1 + II_2. \end{aligned}$$

For II_1 and II_2 , by Hölder's inequality, we obtain

$$\begin{aligned} II_1 &\leq C p^{\frac{-krn}{r'}} \left\{ \left(\sum_{j=-\infty}^k \int_{S_j} |f(t)|^r dt \right)^{\frac{1}{r}} \left(\int_{S_j} |b(t) - b_{B_j}|^{r'} dt \right)^{\frac{1}{r'}} \right\}^r \\ &\leq C p^{\frac{-krn}{r'}} \left\{ \sum_{j=-\infty}^k \|f_j\|_{L^r(\mathbb{Q}_p^n)} p^{\frac{jn}{r'}} \left(\frac{1}{|B_j|_H} \int_{B_j} |b(t) - b_{B_j}|^{r'} dt \right)^{\frac{1}{r'}} \right\}^r \quad (3.6) \\ &\leq C \|b\|_{CMO^{r'}(\mathbb{Q}_p^n)}^r \left\{ \sum_{j=-\infty}^k p^{\frac{(j-k)n}{r'}} \|f_j\|_{L^r(\mathbb{Q}_p^n)} \right\}^r. \end{aligned}$$

And

$$\begin{aligned} II_2 &\leq C p^{\frac{-krn}{r'}} \|b\|_{CMO^1(\mathbb{Q}_p^n)}^r \left\{ \sum_{j=-\infty}^k (k-j) \left(\int_{S_j} |f(t)|^r dt \right)^{\frac{1}{r}} \left(\int_{S_j} dt \right)^{\frac{1}{r'}} \right\}^r \\ &\leq C \|b\|_{CMO^1(\mathbb{Q}_p^n)}^r \left\{ \sum_{j=-\infty}^k (k-j) p^{\frac{(j-k)n}{r'}} \|f_j\|_{L^r(\mathbb{Q}_p^n)} \right\}^r. \quad (3.7) \end{aligned}$$

Then (3.5)–(3.7) together with Lemma 3.1 imply that

$$\|\mathcal{H}_b f\|_{K_r^{\alpha, q_2}(\mathbb{Q}_p^n)} = \left(\sum_{k=-\infty}^{+\infty} p^{k\alpha q_2} \|(U_{\beta, b} f) \chi_k\|_{L^r(\mathbb{Q}_p^n)}^{q_2} \right)^{\frac{1}{q_2}}$$

$$\begin{aligned}
&\leq \left(\sum_{k=-\infty}^{+\infty} p^{k\alpha q_1} \|(U_{\beta,b}f)\chi_k\|_{L^r(\mathbb{Q}_p^n)}^{q_1} \right)^{\frac{1}{q_1}} \\
&\leq C \left(\sum_{k=-\infty}^{+\infty} p^{k\alpha q_1} \|b\|_{CMO^r(\mathbb{Q}_p^n)}^{q_1} \left(\sum_{j=-\infty}^k p^{\frac{(j-k)n}{r'}} \|f_j\|_{L^r(\mathbb{Q}_p^n)} \right)^{q_1} \right)^{\frac{1}{q_1}} \\
&\quad + C \left(\sum_{k=-\infty}^{+\infty} p^{k\alpha q_1} \|b\|_{CMO^{r'}(\mathbb{Q}_p^n)}^{q_1} \left(\sum_{j=-\infty}^k p^{\frac{(j-k)n}{r'}} \|f_j\|_{L^r(\mathbb{Q}_p^n)} \right)^{q_1} \right)^{\frac{1}{q_1}} \\
&\quad + C \left(\sum_{k=-\infty}^{+\infty} p^{k\alpha q_1} \|b\|_{CMO^1(\mathbb{Q}_p^n)}^{q_1} \left(\sum_{j=-\infty}^k (k-j) p^{\frac{(j-k)n}{r'}} \|f_j\|_{L^r(\mathbb{Q}_p^n)} \right)^{q_1} \right)^{\frac{1}{q_1}} \\
&\leq C \|b\|_{CMO^{\max\{r',r\}}(\mathbb{Q}_p^n)} \left(\sum_{k=-\infty}^{+\infty} p^{k\alpha q_1} \left(\sum_{j=-\infty}^k (k-j) p^{\frac{(j-k)n}{r'}} \|f_j\|_{L^r(\mathbb{Q}_p^n)} \right)^{q_1} \right)^{\frac{1}{q_1}} \\
&:= J.
\end{aligned} \tag{3.8}$$

For the case $0 < q_1 \leq 1$, since $\alpha < \frac{n}{r'}$, we have

$$\begin{aligned}
J^{q_1} &= C \|b\|_{CMO^{\max\{r',r\}}(\mathbb{Q}_p^n)}^{q_1} \sum_{k=-\infty}^{+\infty} p^{k\alpha q_1} \left(\sum_{j=-\infty}^k (k-j) p^{\frac{(j-k)n}{r'}} \|f_j\|_{L^r(\mathbb{Q}_p^n)} \right)^{q_1} \\
&= C \|b\|_{CMO^{\max\{r',r\}}(\mathbb{Q}_p^n)}^{q_1} \sum_{k=-\infty}^{+\infty} \left(\sum_{j=-\infty}^k p^{j\alpha} \|f_j\|_{L^r(\mathbb{Q}_p^n)} (k-j) p^{(j-k)(\frac{n}{r'}-\alpha)} \right)^{q_1} \\
&\leq C \|b\|_{CMO^{\max\{r',r\}}(\mathbb{Q}_p^n)}^{q_1} \sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^k p^{j\alpha q_1} \|f_j\|_{L^r(\mathbb{Q}_p^n)}^{q_1} (k-j)^{q_1} p^{(j-k)(\frac{n}{r'}-\alpha)q_1} \\
&= C \|b\|_{CMO^{\max\{r',r\}}(\mathbb{Q}_p^n)}^{q_1} \sum_{j=-\infty}^{+\infty} p^{j\alpha q_1} \|f_j\|_{L^r(\mathbb{Q}_p^n)}^{q_1} \sum_{k=j}^{+\infty} (k-j)^{q_1} p^{(j-k)(\frac{n}{r'}-\alpha)q_1} \\
&= C \|b\|_{CMO^{\max\{r',r\}}(\mathbb{Q}_p^n)}^{q_1} \|f\|_{K_r^{\alpha,r}(\mathbb{Q}_p^n)}^{q_1}.
\end{aligned} \tag{3.9}$$

For the case $q_1 > 1$, by Hölder's inequality, we have

$$\begin{aligned}
J^{q_1} &\leq C \|b\|_{CMO^{\max\{r',r\}}(\mathbb{Q}_p^n)}^{q_1} \sum_{k=-\infty}^{+\infty} \left(\sum_{j=-\infty}^k p^{j\alpha q_1} \|f_j\|_{L^r(\mathbb{Q}_p^n)}^{q_1} p^{\frac{(j-k)n}{2}(\frac{n}{r'}-\alpha)q_1} \right) \times \\
&\quad \left(\sum_{j=-\infty}^k (k-j)^{q_1'} p^{\frac{(j-k)n}{2}(\frac{n}{r'}-\alpha)q_1'} \right)^{\frac{q_1}{q_1'}}
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&= C \|b\|_{CMO^{\max\{r', r\}}(\mathbb{Q}_p^n)}^{q_1} \sum_{j=-\infty}^{+\infty} p^{j\alpha q_1} \|f_j\|_{L^r(\mathbb{Q}_p^n)}^{q_1} \sum_{k=j}^{+\infty} p^{\frac{(j-k)}{2}(\frac{n}{r'}-\alpha)q_1} \\
&= C \|b\|_{CMO^{\max\{r', r\}}(\mathbb{Q}_p^n)}^{q_1} \|f\|_{K_r^{\alpha, r}(\mathbb{Q}_p^n)}^{q_1}.
\end{aligned}$$

Then (1) follows from (3.8), (3.9) and (3). By the similar method, we can get (2).

Theorem 3.1 is proved. \square

Proof of Theorem 3.2. By definition, we have

$$\begin{aligned}
|T_b^p f| &= \left| \int_{\mathbb{Q}_p^*} \frac{f(y)}{\max(|x|_p, |y|_p)} (b(x) - b(y)) dy \right| \\
&\leq \left| \int_{B(0, |x|_p)} \frac{f(y)}{|x|_p} (b(x) - b(y)) dy \right| + \left| \int_{\mathbb{Q}_p^n \setminus B(0, |x|_p)} \frac{f(y)}{|y|_p} (b(x) - b(y)) dy \right| \quad (3.11) \\
&= |\mathcal{H}_b^p f| + |\mathcal{H}_b^{p,*} f|.
\end{aligned}$$

By Minkowski's inequality, we get

$$\|T_b^p f\|_{K_r^{\alpha, q_2}(\mathbb{Q}_p^n)} \leq \|\mathcal{H}_b^p f\|_{K_r^{\alpha, q_2}(\mathbb{Q}_p^n)} + \|\mathcal{H}_b^{p,*} f\|_{K_r^{\alpha, q_2}(\mathbb{Q}_p^n)}. \quad (3.12)$$

Then Theorem 3.2 follows from Theorem 3.1. \square

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